International Research Journal of Education and Technology (IRJEdT)
║ ISSN: 2581-7795, Impact Factor 5.007 ║
║ Peer-Reviewed | Open Access | International Journal ║
║ Volume: 09 | Issue: 08 | Aug- 2026 | www.irjweb.com ║

Patterns of Integer Solutions to Homogeneous Ternary Quadratic Surface $3(x^2+y^2)-5xy=36z^2$

V.Anbuvalli[1], J.Shanthi[2], M.A.Gopalan[3]

[1] Assistant Professor, Department of Mathematics,

Srimati Indira Gandhi College ,

Affiliated to Bharathidasan University,

Tiruchirappalli-620 002, Tamil Nadu, India.

https://orcid.org/0009-0006-0905-9123

[2]Assistant Professor and Head, Department of Mathematics,

Shrimati Indira Gandhi College,

Affiliated to Bharathidasan University,

Tiruchirappalli-620002, Tamil Nadu, India.

https://orcid.org/0009-0008-5945-410X

[3]Professor, Department of Mathematics, Shrimati Indira Gandhi College,
Affiliated to Bharathidasan University, Tiruchirappalli-620002, Tamil Nadu, India.
https://orcid.org/0000-0003-1307-2348

Abstract:

The cone represented by the ternary quadratic Diophantine Equation $3(x^2+y^2)-5xy=36z^2$ is analyzed for its patterns of non-zero distinct integral solutions. Substitution technique and factorization method are utilized to obtain the same.

Keywords: Ternary quadratic equation, Cone, Integral solutions, Substitution technique, Factorization method

Introduction:

The Diophantine equation offers an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-11] for quadratic equations with multiple variables. This communication concerns with yet another interesting equation $3(x^2+y^2)-5xy=36z^2$ representing homogeneous quadratic with three unknowns for determining its infinitely many non-zero integral points.

Method of analysis:

The homogeneous ternary quadratic Diophantine equation under considerable is

$$3(x^2+y^2)-5xy=36z^2 \tag{1}$$

Choosing

$$x=6X,\ y=6Y \tag{2}$$

in (1), it is written as

$$3(X^2+Y^2)-5XY=36Z^2 \tag{3}$$

Introduction of the linear transformations

$$X=u+v, Y=u-v, u\neq \pm v\neq 0 \tag{4}$$

in (3) gives

$$u^2+11v^2=z^2 \tag{5}$$

The above equation (5) is solved through different ways to obtain $u, v \,\&\, z$.

In view of (4) and (2), the corresponding values of x, y satisfying (1) are obtained.

Way 1:

Let

$$z=z(a,b)=a^2+11b^2 \tag{6}$$

Using (6) in (5). Employing factorization and equating the positive factors, one has

$$u + i\sqrt{11}\,v = (a + i\sqrt{11}b)^{2}$$

$$= f(a,b) + i\sqrt{11}g(a,b) \quad \textit{where} \quad f(a,b) = a^{2} - 11b^{2}$$
$$g(a,b) = 2ab$$

Equating the real and imaginary in the above equation, we get

$$u = f(a,b),\ v = g(a,b)$$

From (4) and (2), one has

$$x = 6X = 6(u+v) = 6(f(a,b) + g(a,b))$$
$$y = 6Y = 6(u-v) = 6(f(a,b) - g(a,b))$$

Thus, (7) and (6) satisfy (1)

Way 2:

Write (5) as

$$u^{2} + 11v^{2} = z^{2} * 1 \tag{8}$$

The integer 1 in (8) is written as the product of complex conjugates as below

$$1 = \frac{(5 + i\sqrt{11})\ (5 - i\sqrt{11})}{36} \tag{9}$$

Substitute (6) and (9) in (8). Employing factorization and equating

positive factors , one has

$$u + i\sqrt{11}v = \frac{(5 + i\sqrt{11})}{6}(a + i\sqrt{11}b)^{2}$$
$$= \frac{1}{6}(5 + i\sqrt{11})\,(f(a,b) + i\sqrt{11}g(a,b)\ )$$

Equating the real and imaginary in the above equation, we get

$$u = \frac{1}{6}\left[5f(a,b) - 11g(a,b)\right]$$
$$v = \frac{1}{6}\left[f(a,b) + 5g(a,b)\right]$$

Now from (4) and (2), we have

$$\begin{aligned} x &= 6X = 6(u+v) = 6f(a,b) - 6g(a,b) \\ y &= 6Y = 6(u-v) = 4f(a,b) - 16g(a,b)) \end{aligned} \quad (10)$$

Thus, (10) and (6) satisfy (1).

Note: 1

The integer 1 on the R.H.S of (8) may be written as the product of complex conjugates as shown below

$$1 = \frac{(1+i3\sqrt{11})\ (1-i3\sqrt{11})}{100}$$
$$1 = \frac{(11r^2 - s^2 + i\sqrt{11}(2rs))(11r^2 - s^2 - i\sqrt{11}(2rs))}{(11r^2+s^2)^2}$$
$$1 = \frac{(r^2 - 11s^2 + i\sqrt{11}(2rs))(r^2 - 11s^2 - i\sqrt{11}(2rs))}{(r^2+11s^2)^2}$$
$$1 = \frac{(11s^2 - 1 + i\sqrt{11}(2s))(11s^2 - 1 - i\sqrt{11}(2s))}{(11s^2+1)^2}$$
$$1 = \frac{(22s^2 - 22s + 5) + i\sqrt{11}(2s-1))(22s^2 - 22s + 5 - i\sqrt{11}(2s-1))}{(22s^2 - 22s + 6)^2}$$

Way 3:

$$z^2 - 11v^2 = u^2 * 1 \quad (11)$$

Assume

$$u = a^2 - 11b^2 = (a + \sqrt{11}b)(a - \sqrt{11}b) \tag{12}$$

The integer 1 in (11) is written as

$$1 = \frac{(6 + i\sqrt{11})\ (6 - \sqrt{11})}{25} \tag{13}$$

Substitute (12) and (13) in (11). Employing factorization and equating

positive factors , one has

$$z + \sqrt{11}\,v = \frac{1}{5}(6 + \sqrt{11})\,(a^2 + 2\sqrt{11}ab + 11b^2$$

Equating the rational and irrationals in the above equation, we get

$$\begin{aligned} z &= \frac{1}{5}\left[6(a^2 + 11b^2) + 22ab\right] \\ v &= \frac{1}{5}\left[(a^2 + 11b^2) + 12ab\right] \end{aligned} \tag{14}$$

As our aim is to obtain integer solutions, replacing a by a by 5A, b by 5B in (12) and (14), one has

$$\begin{aligned} z &= 5\left[6(A^2 + 11B^2) + 22AB\right] \\ v &= 5\left[(A^2 + 11B^2) + 12AB\right] \end{aligned} \tag{15}$$

From (4) and (2), we have

$$\begin{aligned} x &= 6X = 6(u + v) = 6\left[30A^2 - 220B^2 + 60AB\right] \\ y &= 6Y = 6(u - v) = 6\left[20A^2 - 330B^2 - 60AB\right] \end{aligned} \tag{16}$$

Thus, (15) and (16) satisfy (1).

Note: 2

Apart from (13), the integer 1 on the R.H.S of (11) may be written as following:

$$1=\frac{(10+3\sqrt{11})\;(10-3\sqrt{11})}{1}$$

$$1=\frac{(11r^2+s^2+\sqrt{11}(2rs))(11r^2+s^2-\sqrt{11}(2rs))}{(11r^2-s^2)^2}$$

Conclusion:

In this paper, we have made an attempt to obtain all integer solutions to (1). To conclude, one may search for integer solutions to other choices of homogeneous, non-homogeneous cones.

References:

[1] L.E. Dickson, History of Theory of Numbers, Vol 2, Chelsea Publishing Company, New York (1952).

[2] L.J. Mordell, Diophantine Equations, Academic press, London, (1969).

[3] R.D. Carmichael, The theory of numbers and Diophantine analysis, New York, Dover, (1959).

[4] J. Shanthi, M.A. Gopalan, Ternary Homogeneous Quadratic $x^2+y^2=145z^2$, Jnanabha, Vol 55 (1), 125–128, May 2025.

[5] J. Shanthi, M. Parkavi, On Finding Integer Solutions to the Homogeneous Ternary Quadratic Diophantine equation $2(x^2+y^2)-3xy=32z^2$, International Journal of Research Publication and Reviews, 4(1) (2023), 200–208.

[6] Thiruniraiselvi N, Gopalan M. A. A Class of new solutions in integers to Ternary Quadratic Diophantine Equation $12(x^2+y^2)-23xy+2x+2y+4=56z^2$ International Journal of Research Publication and Reviews 2024, 5(5):3224-3226.

[7] Dr. J. Shanthi, Dr. M.A. Gopalan, A Glimpse on Homogeneous Ternary Quadratic Diophantine Equation, International Advanced Research Journal in Science, Engineering and Technology, Volume 11, Issue 9, pp 96–101, September 2024.

[8] M.A. Gopalan, S. Vidhyalakshmi and N. Thiruniraiselvi, Observations on the ternary quadratic Diophantine equation $x^2 + 9y^2 = 50z^2$, International Journal of Applied Research, Vol. 1(2), pp 51–53, 2015.

[9] M.A. Gopalan, S. Vidhyalakshmi, and N. Thiruniraiselvi, On the Homogeneous Ternary Quadratic Diophantine Equation $6x^2 + 5y^2 = 341z^2$, Vidyabharati International Interdisciplinary Research Journal, (Special Issue on Recent Research Trends in Management, Science and Technology – Part-1, pdf page 318), Pg: 1612–1617, 2021.

[10] J. Shanthi, M.A. Gopalan, Observations on the Integer Solutions of the Ternary Quadratic Equation $x^2 + y^2 - z^2 = 10$, International Research Journal of Educational and Technology, Volume 04, Issue 07, 220–231, July 2022.

[11] J. Shanthi, On Finding Integer Solutions to the Homogeneous Ternary Quadratic Diophantine Equation $2(x^2 + y^2) - 3xy = 32z^2$, International Journal of Research Publication and Reviews, Vol 4, no 1, pp 200–208, January 2023.